Waves in anisotropic media

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An important new field of study in geophysical and astrophysical fluid dynamics, with possible repercussions for plasma dynamics, electrodynamics and nanophotonics, is the study of perturbations of anisotropic equilibria.

Examples of anisotropic equilibria from fluid dynamics are density-stratified fluids in a field of gravity, and homogeneous density fluids subjected to uniform rotation (plus any combination of the two). Their perturbations are known as internal gravity and inertial waves respectively.

The equilibria in plasma dynamics, electrodynamics and nanophotonics are less easy to define, but their perturbations are well-known: in plasma dynamics and electrodynamics, these electromagnetic waves, influenced by an external magnetic field, are called 'whistlers' and electron-cyclotron waves respectively.
In *nanophotonics,* anomalous electromagnetic waves occur in *metamaterials* - stacked layers of material with different optical properties (such as permittivity and susceptibility). The anisotropic layering requires light to be confined to nanometer scales, much smaller than their vacuum wavelengths of several hundreds of nanometers.

In all cases, *anisotropy* dominates the nature of the response: anisotropy due to the preferred orientation of gravity, rotation vector, magnetic field or stacking direction of the metamaterial. In a gravitationally stable density stratification, fluid parcels are for instance easily (i.e. 'free of charge') moved within each plane perpendicular to the direction of gravity, while it is energetically costly to move them up or down the direction of gravity.

Interestingly, in each of these media, anisotropy leads to perturbations (waves) that all obey an anomalous dispersion relation in which wave frequency relates to wave vector *inclination* relative to the anisotropy direction, rather than to wave vector *magnitude,* as is commonly the case in isotropic media. Its consequences for the perturbations are profound. For a symmetry-breaking shape of the cavity (i.e. a cavity having at least one boundary oriented neither parallel nor perpendicular to the anisotropy direction), it leads to *multiscale* behavior of *linear* wave solutions, a feature otherwise found only in the manifestly nonlinear field of turbulence. However, it should be mentioned that this property pertains to spatial scales only and not, as for turbulence, to time scales as well. These results are found for small-amplitude *perturbations* and not, as for turbulence, just at a *nonlinear* level for large-amplitude perturbations. This implies that conditions on the magnitude of the disturbance are much milder than required for invoking a nonlinear response. Quite generically, the waves will approach a *wave-attractor* - a highly predictable and persistent location where amplified motions and exchange processes will take place, regardless of the precise location inside the cavity where these perturbations originate from [1,2].

Localization of wave energy on an attractor is interesting as its multiscale property signifies a natural route between the largest scale, at which energy is usually entering a system coherently, towards the smallest scales, where energy is dissipated. The large-scale structure is often associated with a slow manifold, which is typically modelled as being disjoint from the fast (i.e. small scale) manifold, at which dissipation occurs. By contrast, the simultaneous presence of these two disparate scales is found to be an integral property of a wave attractor. Perhaps more remarkable is the fact that while the underlying dynamical problems are all linear -- after all they describe *perturbations* to an equilibrium state --, the resulting wave fields have properties normally associated with *nonlinear dynamics* (such as *selfsimilarity* in physical, Fourier and parameter spaces). There is however good reason for the presence of nonlinear features inside a dynamically linear problem setting: the path of characteristics that are employed in solving these linear anisotropic problems is based on a nonlinear map of the boundary onto itself. Thus, the nonlinear shape of the cavity boundary determines the presence of attracting orbits (Figs. 1,2) and of the selfsimilar shape of the wave field [2,3].

The study of waves in anisotropic media invites further investigation in future studies as it may have applications in oceanographical, meteorological and also in planetary and stellar exchange processes. To illustrate this in the latter context, it is remarkable that lithium depletion was found to occur strictly in stars that are accompanied by big planets. The implied presence of tidal forces in the rotating and stratified stellar atmosphere can tangibly be expected to open a venue for rapid chemical transport via the periodic bands constructed by wave attractors.
Similarly, wave attractors are speculated to be relevant to the presence of striped patterns on the surface of icy moons, as Saturn's moon Enceladus. These '*tiger stripes*' possibly betray the action of sustained shearing motion onto the ice cover, produced by focused internal gravity or inertial waves in an underlying ocean [4].

This topic may also be of interest to industrial applications involving rotating fluids and gasses in rotating or revolving machinery. For instance, we can speculate that wave attractors might be relevant in bioreactors, which consist of fluids containing several species of algae and nutrients. These species, that may produce new pharmaceutical chemicals/food, are however difficult to keep in cultures because of conflicting demands on mixing. Fluid inside such a reactor needs to be stirred sufficiently strong so as to bring algae into contact with dissolved nutrients, yet to do this gently enough so as to not destroy fragile exotic algae species, due to excessive shearing motion.
Likewise, the presence of wave attractors may potentially have an application for light trapping in metamaterials and hence to a new line of data storage and data retrieval in chip-industry.

To return to geophysical fluids, the topic of wave structures produced by anisotropy is of direct interest to oceanography - a stable and continuously-stratified fluid - as the concept of wave-attractors offers a paradigm in which an amplified, localized isopycnal (an iso-density pleth) as well as cross-isopycnal transport of heat, momentum, oxygen, plankton and nutrients can possibly be rationalized. As such, this is not only of academic interest, but also of concern to deep-sea life and deep-sea mining activities. Its detection in the ocean is however presently hampered by lack of subsurface, spatially-resolving instruments. Given that theoretical and consonant laboratory experimental [2] and numerical [1] work all suggest the existence of wave attractors in the sea, the developments of sensors that will be able to resolve the spatial structure of the internal wave field at sea, should be promoted.

[1] Hamiltonian discontinuous Galerkin FEM for linear, stratified (in)compressible Euler equations: internal gravity waves. A.M. van Oers, L.R.M. Maas, O. Bokhove (2017) Journal of Computational Physics 330, 770-793

[2] Internal wave attractors in three-dimensional geometries: trapping by oblique reflection. G. Pillet, E. V. Ermanyuk, L. R. M. Maas, I. N. Sibgatullin and T. Dauxois. *Journal of Fluid Mechanics* (2018), vol. 845, pp. 203-225.

[3] Internal Wave Attractors in 3D Geometries: a dynamical systems approach. G. Pillet, L. R. M. Maas and T. Dauxois. (2018) Submitted to *European Journal of Mechanics - B/Fluids.*

[4] Do tidally-generated inertial waves heat the subsurface oceans of Europa and Enceladus? M. Rovira-Navarro, M. Rieutord, T. Gerkema, L.R.M. Maas, W. van der Wal, B. Vermeersen. (2018) Submitted to *Icarus.*

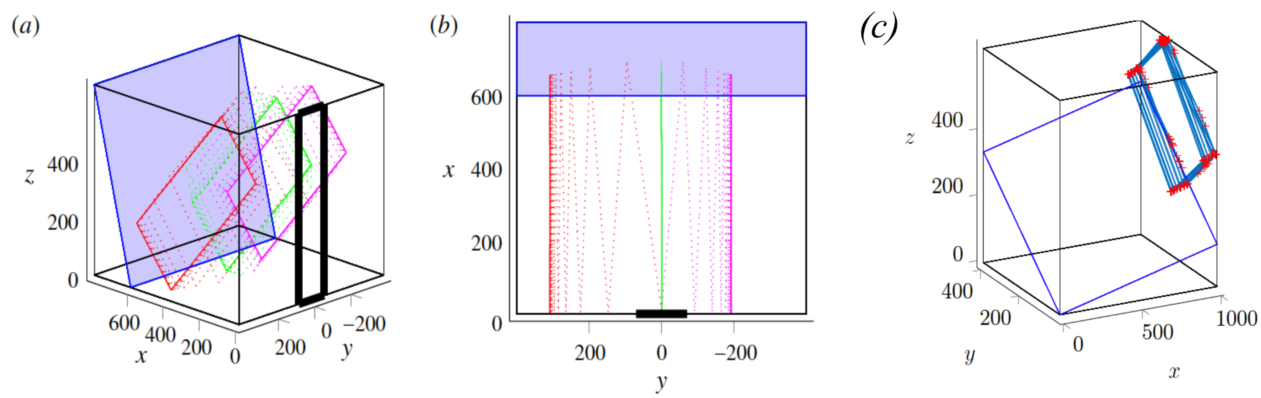

Fig. 1 Uniformly-stratified basin, having one side sloping (blue) in a quasi-2D (a,b) or truly 3D setting (c). Internal waves are forced in black square region, seen in perspective (a) and from above (b). When reflecting from the slope, internal gravity waves get focused and diffract towards a direction normal to the slope, ultimately being trapped on a wave attractor (solid red, green and purple lines) in a single perpendicular plane, whose location is determined by the initial horizontal launching direction. Together these build a two-dimensional attracting manifold [2]. (c) In 3D, the waves focus onto a nearly one-dimensional attracting manifold: a ‘superattractor’ [3].

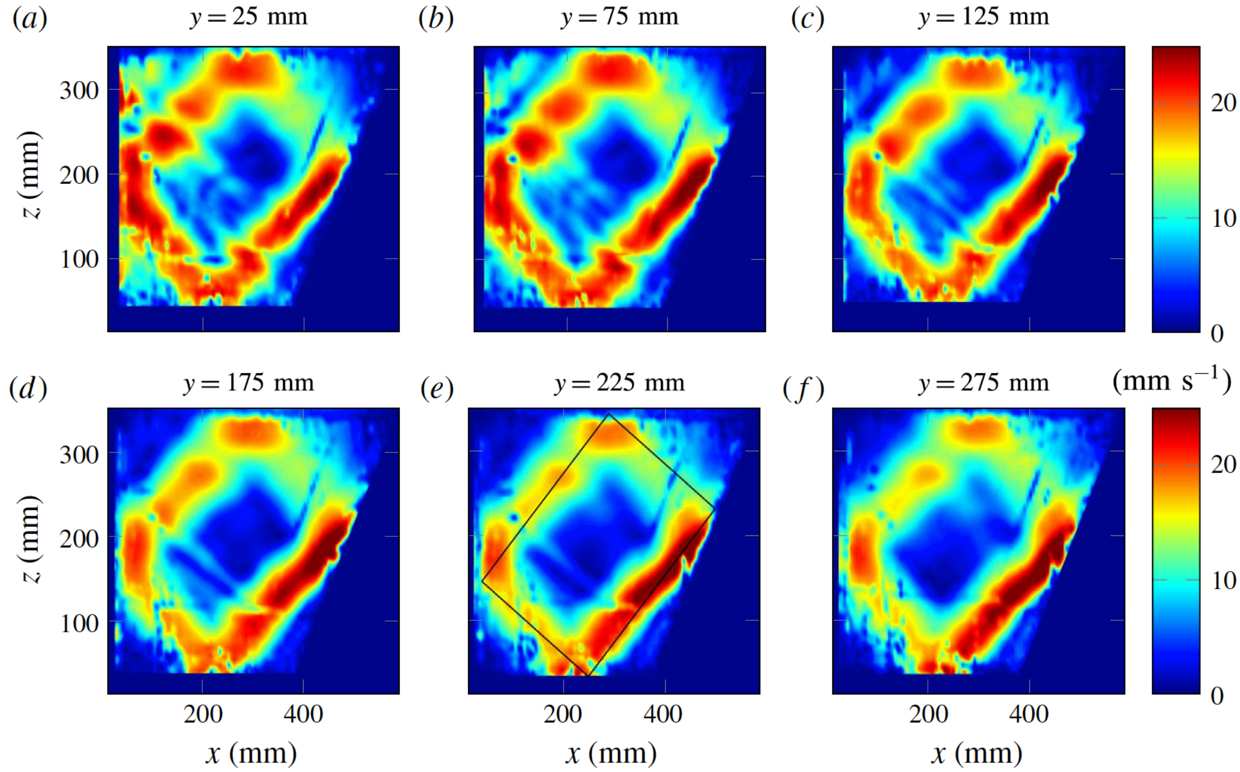

Fig. 2 Experimental manifestation of wave attractors in a uniformly-stratified fluid contained in a tank with one sloping wall as in Fig. 1a,b. (a-f) show magnitude of velocity field in cross-sections occurring at several transverse y-positions. Waves are forced as shown, in Fig. 1a,b, only over the range -60mm < y < 60mm. Wave diffraction, occurring upon focusing reflection, guarantees wave spreading and wave capture onto wave attractors in planes that are oriented perpendicular to the slope: [2], experiments by Grimaud Pillet.