http://acm.org http://ieee.org

Speed-up Solving Linear Systems on Parallel Architectures via Aggregation of Clans

Dmitry A. Zaitsev

http://daze.ho.ua

System of linear algebraic equations

$$\begin{cases} 2x_1 + 5x_2 + 8x_3 = 13 \\ 3x_1 - 6x_2 + 9x_3 = 25 \\ 4x_1 + 7x_2 - 1x_3 = -7 \end{cases}$$

$$x_1 \approx 0,302; \quad x_2 \approx -0,873;$$

$$x_3 \approx 2,095$$

Matrix representation

$$A\vec{x} = \vec{b}$$

$$A = \begin{pmatrix} 2 & 5 & 8 \\ 3 & -6 & 9 \\ 4 & 7 & -1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 13 \\ 25 \\ -7 \end{pmatrix}$$

Solution of linear systems

Homogenous:

$$A\vec{x} = 0,\ \vec{x} = G\vec{y}$$

Heterogeneous:

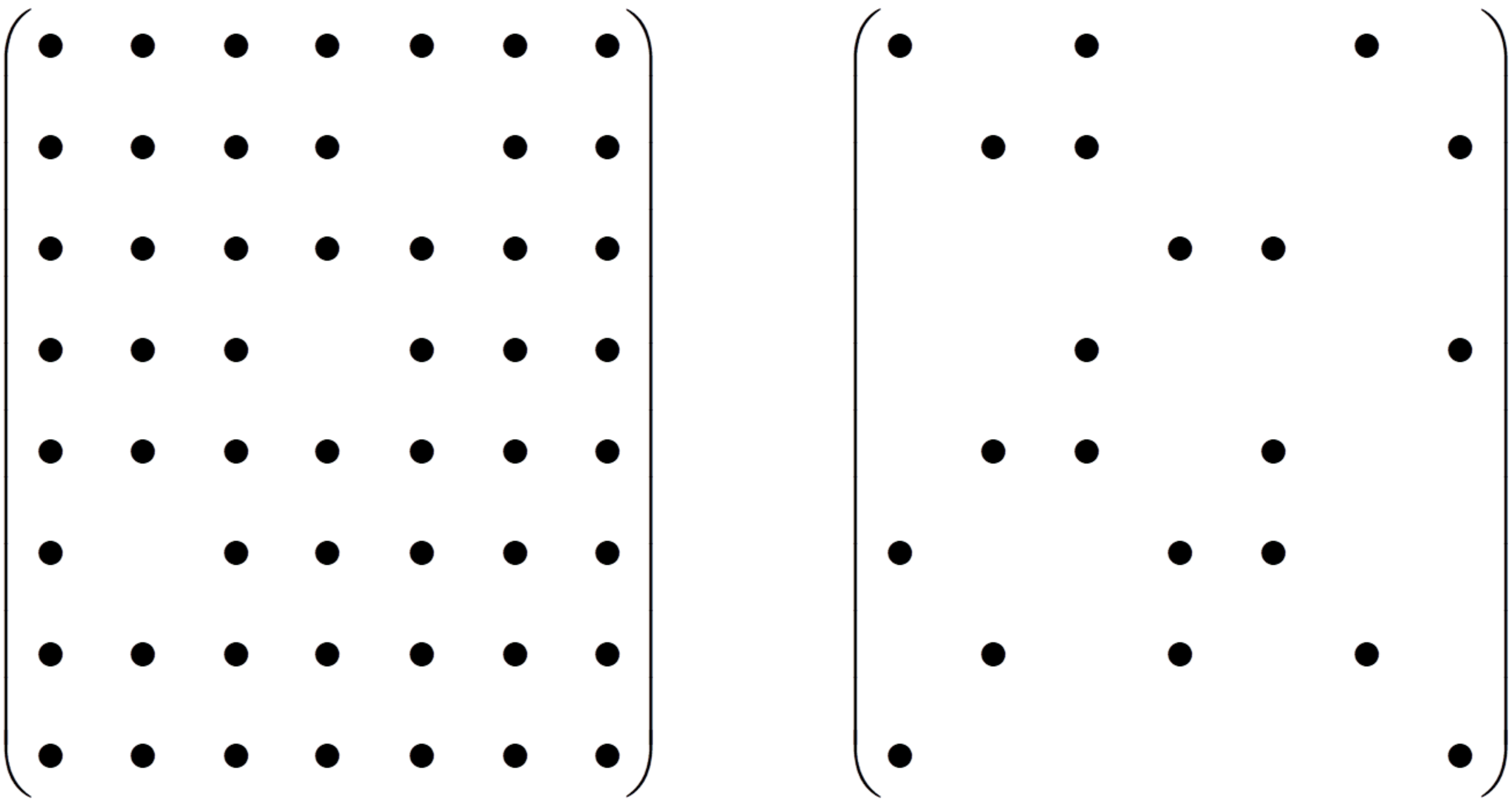
$$A\vec{x} = \vec{b},\ \vec{x} = \vec{x}' + G\vec{y}$$

G - matrix of basis solutions

Algebraic structure

Numbers	Example	Structure	Methods
Complex	-3,2+6,25i	field	a) reduction: LU, QR; б) iteration methods
Real	0,25; -78,931		
Integer	-33; 0; 6	ring	Normal forms: Hermite, Smith
Nonnegative integer	0; 7; 55	monoid	Methods of Toudic (Silva) and Contejean

Dense and sparse systems



Practical value

- Numerical methods of solving differential equations and systems and also partial differential equations and systems
- Domains: thermodynamics, weather forecast, trajectories of moving objects, fluid dynamics, nuclear physics etc
- Diophantine systems: artificial intelligence, cryptography, model-checking etc

Real-life matrices (systems)

- Matrix Market - https://math.nist.gov/MatrixMarket
- The SuiteSparse Matrix Collection https://sparse.tamu.edu
- Model Checking Contest – Petri net models https://mcc.lip6.fr

Basic software

Structure \ Type	Dense	Sparse
Field	LAPACK	UMFPACK
Ring	4ti2	ParAd
Monoid	4ti2	ParAd

LAPACK (**L**inear **A**lgebra **Pack**age)

- http://www.netlib.org/lapack
- Legend of Jack Dongarra (1980) http://icl.utk.edu
- Most widespread package
- Most cited computer science paper
- Performance test of supercomputers http://top500.org

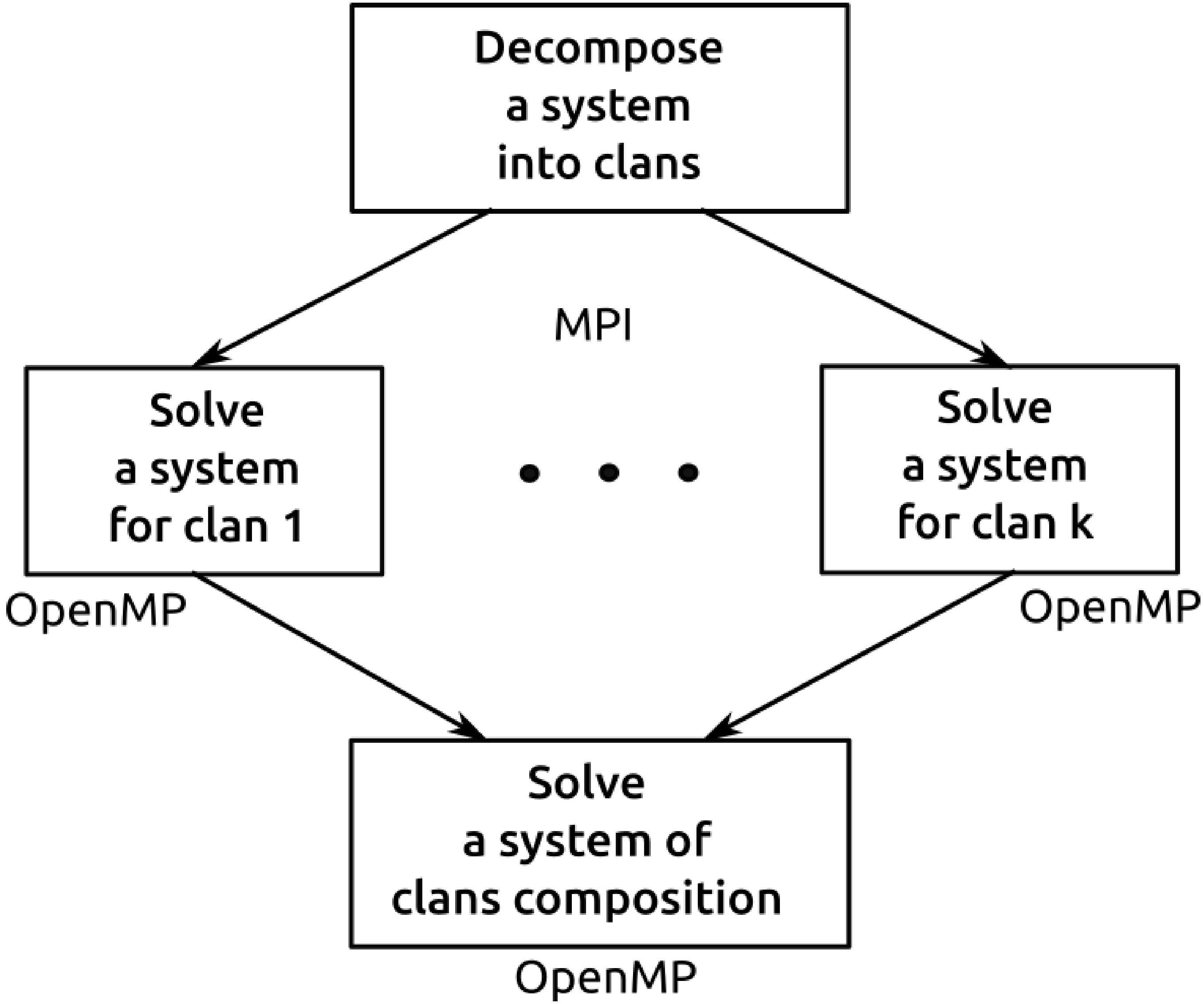
LAPACK on parallel architecture

Multicore	Distributed nodes	GPUs
OpenMP	MPI, PVM	CUDA
PLASMA	ScaLAPACK DPLASMA	MAGMA
http://icl.utk.edu		

Zaitsev decomposition into clans

$$A = \left\| \begin{matrix} A^{0,1} & \widehat{A}^{1} & 0 & 0 & 0 \\ A^{0,2} & 0 & \widehat{A}^{2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{0,k} & 0 & 0 & 0 & \widehat{A}^{k} \end{matrix} \right\|$$

Divide and sway



A Clan – transitive closure of nearness relation

$$
\mathrm{C1:}\ \begin{cases}
-x_2 + x_3 - x_{15} \underline{+ x_{18}} = 0 \\
-x_2 + x_4 - x_{14} \underline{+ x_{18}} = 0 \\
-x_5 + x_6 - x_{16} \underline{+ x_{18}} = 0 \\
-x_{11} + x_{12} - x_{15} \underline{+ x_{18}} = 0 \\
-x_{13} + x_8 \underline{+ x_{18}} = 0
\end{cases}
$$

Two equations are ***near*** if they contain the same variable having coefficients of the same sign

Decomposition into clans

$$\begin{cases} -x_1+x_2-x_{18}=0 \\ -x_2+x_3-x_{15}+x_{18}=0 \\ -x_2+x_4-x_{14}+x_{18}=0 \\ -x_4+x_5+x_{14}-x_{18}=0 \\ -x_3+x_5+x_{15}-x_{18}=0 \\ -x_5+x_6-x_{16}+x_{18}=0 \\ -x_6+x_7+x_{16}-x_{19}=0 \\ x_1-x_7+x_{19}=0 \\ -x_8+x_9-x_{19}=0 \\ -x_9+x_{10}-x_{17}+x_{19}=0 \\ -x_{10}+x_{11}+x_{17}-x_{18}=0 \\ -x_{11}+x_{12}-x_{15}+x_{18}=0 \\ -x_{12}+x_{13}+x_{15}-x_{18}=0 \\ -x_{13}+x_8+x_{18}=0 \end{cases}$$

$$\text{C1:} \begin{cases} -x_2+x_3-x_{15}+x_{18}=0 \\ -x_2+x_4-x_{14}+x_{18}=0 \\ -x_5+x_6-x_{16}+x_{18}=0 \\ -x_{11}+x_{12}-x_{15}+x_{18}=0 \\ -x_{13}+x_8+x_{18}=0 \end{cases}$$

$$\text{C2:} \begin{cases} -x_1+x_2-x_{18}=0 \\ -x_4+x_5+x_{14}-x_{18}=0 \\ -x_3+x_5+x_{15}-x_{18}=0 \\ -x_{10}+x_{11}+x_{17}-x_{18}=0 \\ -x_{12}+x_{13}+x_{15}-x_{18}=0 \end{cases}$$

$$\text{C3:} \begin{cases} x_1-x_7+x_{19}=0 \\ -x_9+x_{10}-x_{17}+x_{19}=0 \end{cases}$$

$$\text{C4:} \begin{cases} -x_6+x_7+x_{16}-x_{19}=0 \\ -x_8+x_9-x_{19}=0 \end{cases}$$

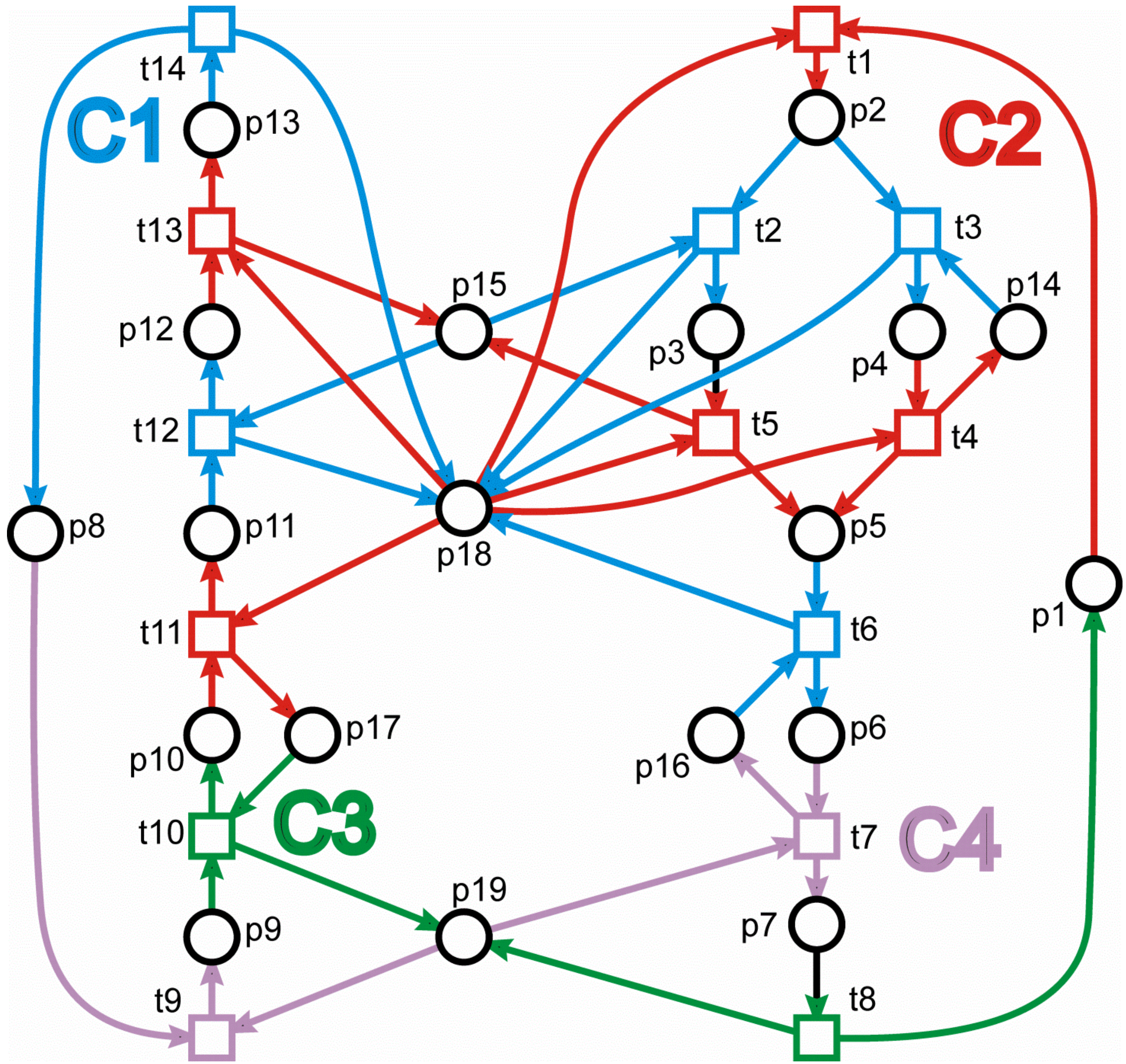
Systems and Directed bipartite graphs

Equation – transition (rectangle)

Variable – place (circle)

Positive sign – incoming arc of a place

Negative sign – outgoing arc of a place



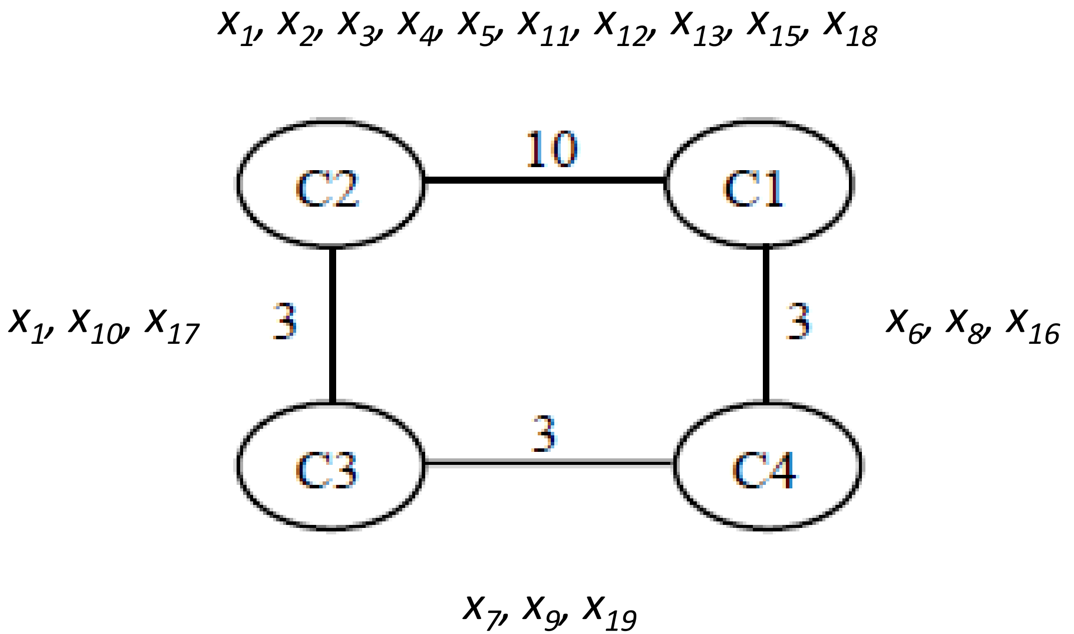
Decomposition graph

$$C1:\begin{cases}-x_2+x_3-x_{15}+x_{18}=0\\ -x_2+x_4-x_{14}+x_{18}=0\\ -x_5+x_6-x_{16}+x_{18}=0\\ -x_{11}+x_{12}-x_{15}+x_{18}=0\\ -x_{13}+x_8+x_{18}=0\end{cases}$$

$$C2:\begin{cases}-x_1+x_2-x_{18}=0\\ -x_4+x_5+x_{14}-x_{18}=0\\ -x_3+x_5+x_{15}-x_{18}=0\\ -x_{10}+x_{11}+x_{17}-x_{18}=0\\ -x_{12}+x_{13}+x_{15}-x_{18}=0\end{cases}$$

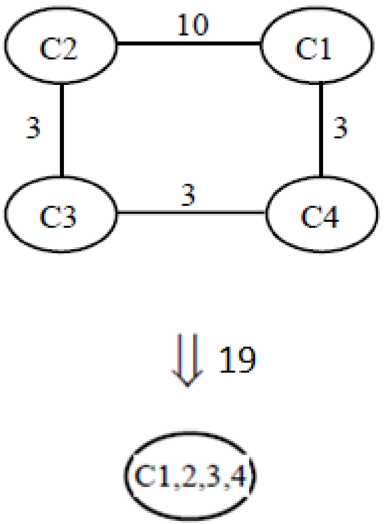
$$C3:\begin{cases}x_1-x_7+x_{19}=0\\ -x_9+x_{10}-x_{17}+x_{19}=0\end{cases}$$

$$C4:\begin{cases}-x_6+x_7+x_{16}-x_{19}=0\\ -x_8+x_9-x_{19}=0\end{cases}$$

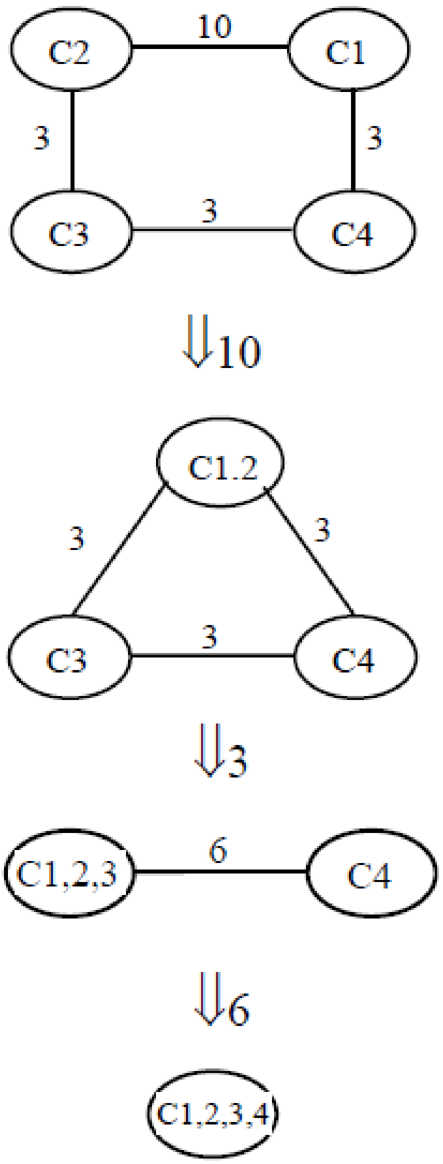


Collapse of decomposition graph

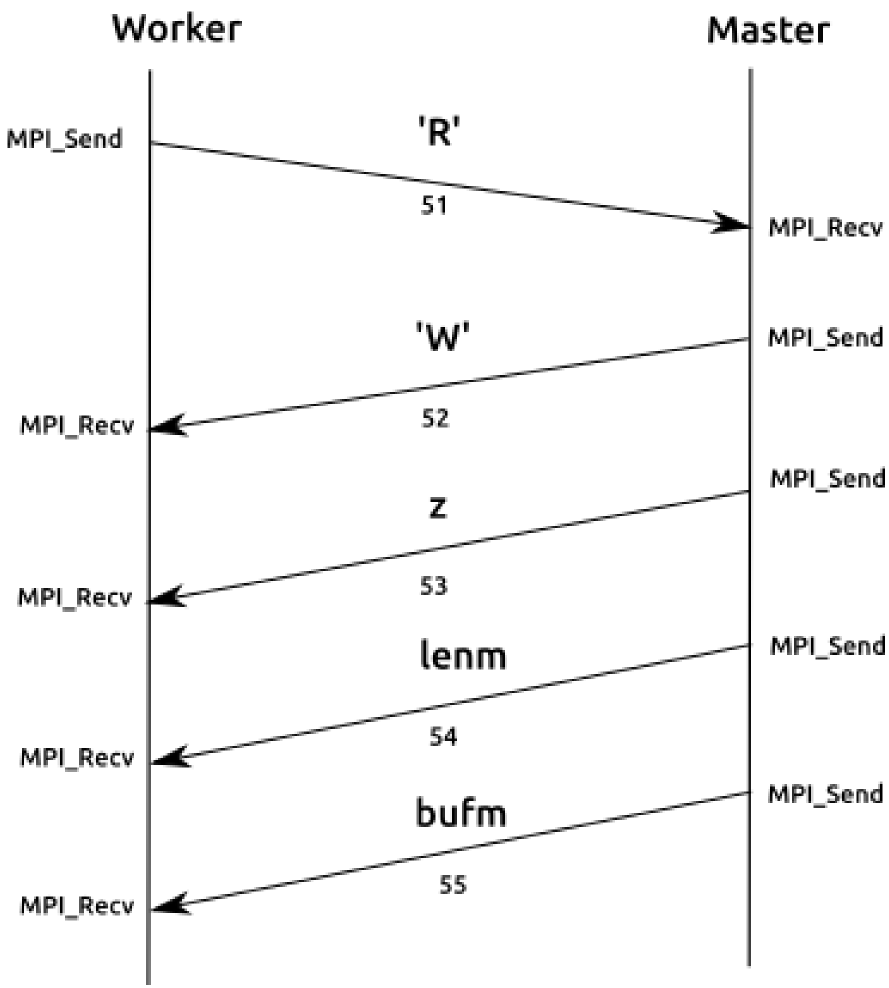
I.

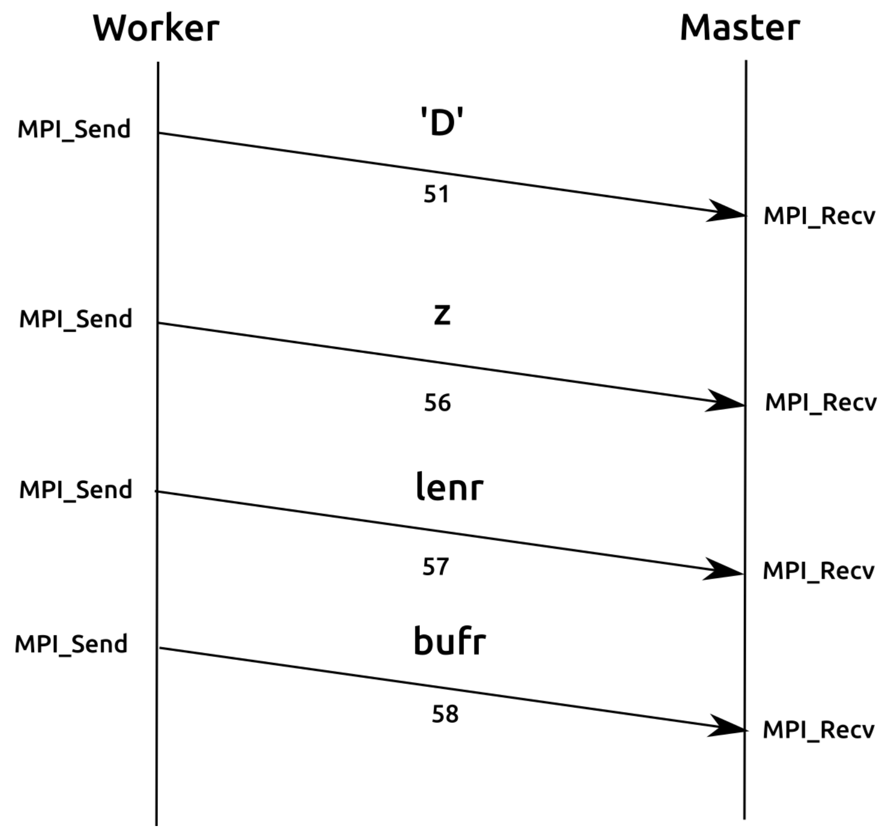


II.

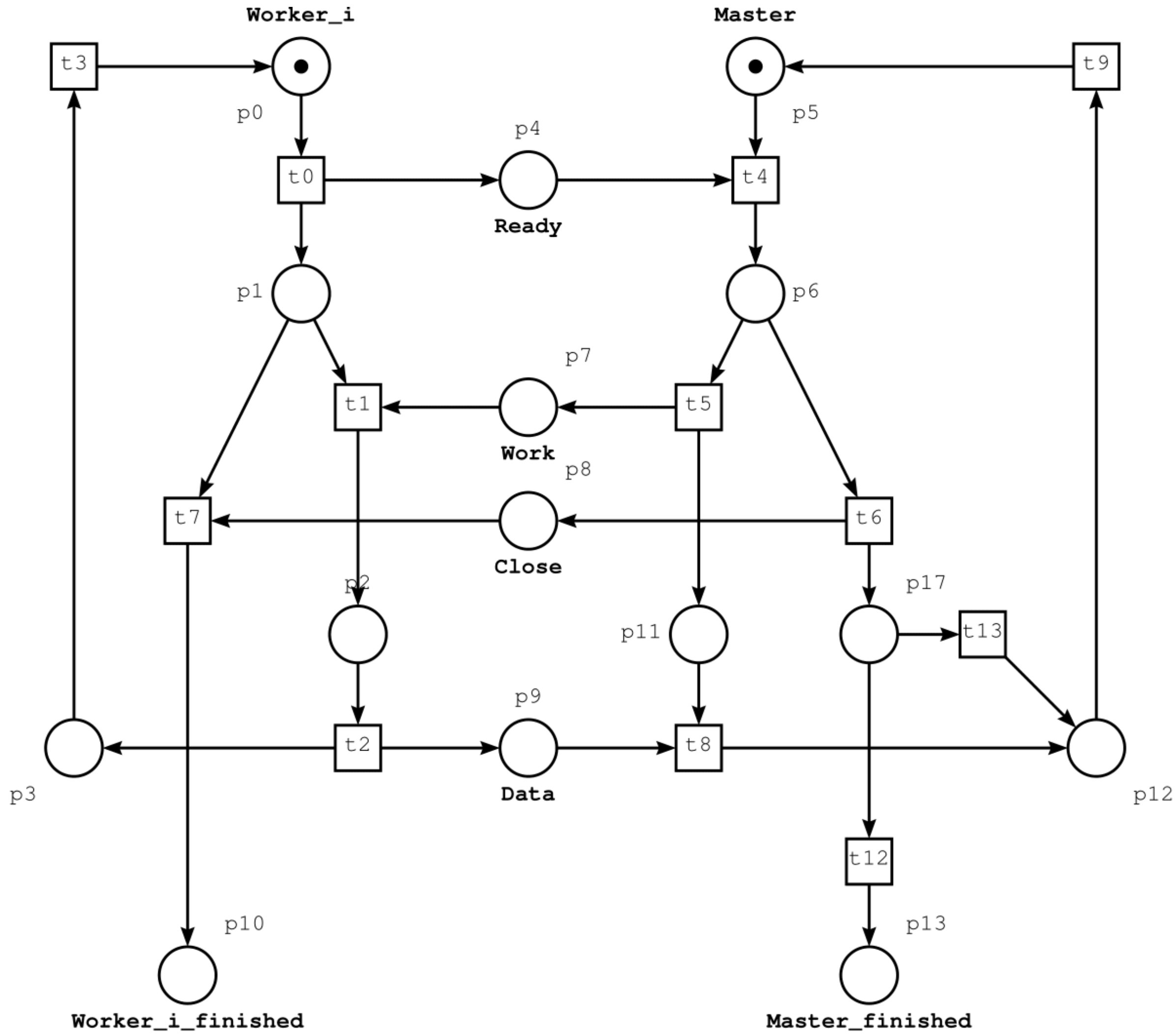


Protocols of data transmission

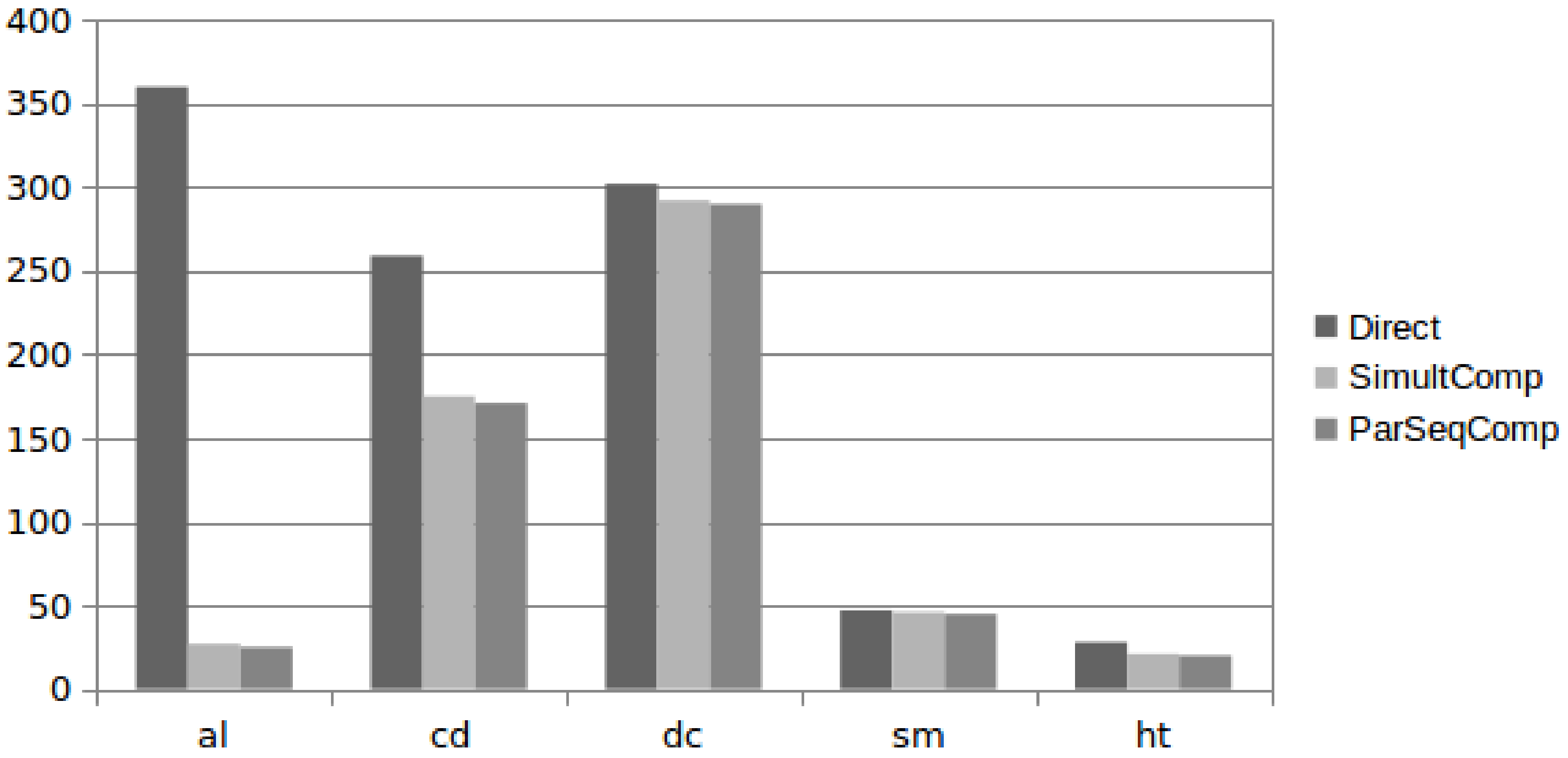




Protocol as Petri net



Speed-up because of clans



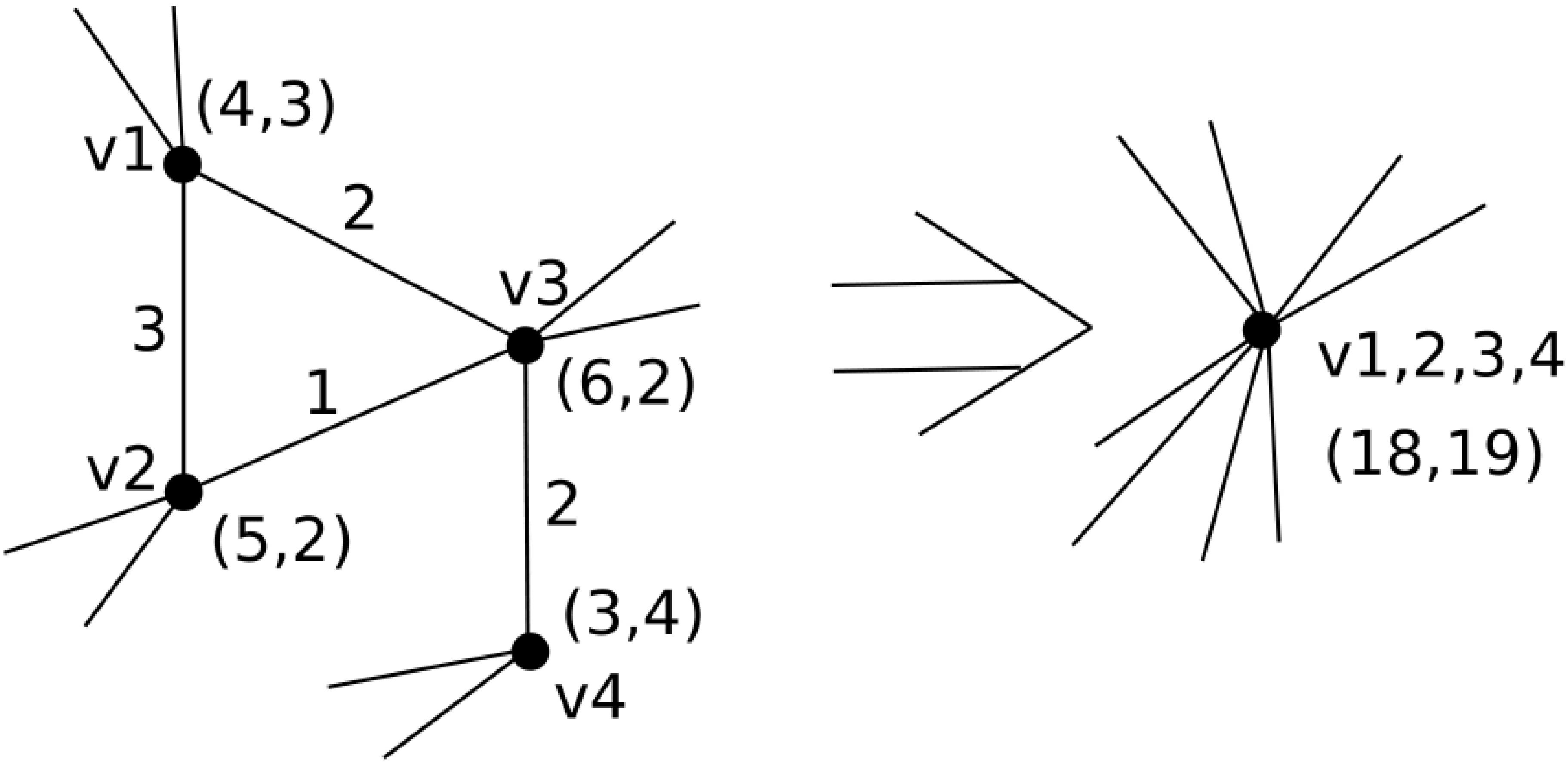
Software

- **Deborah** – decomposition into clans, 2005
- **Adriana** – solving a homogenous system via (a) simultaneous or (b) sequential composition of clans, 2006
- Implemented as plug-ins for Tina http://www.laas.fr/tina
- **ParAd** (Parallel Adriana), 2018 http://github.com/dazeorgacm/ParAd

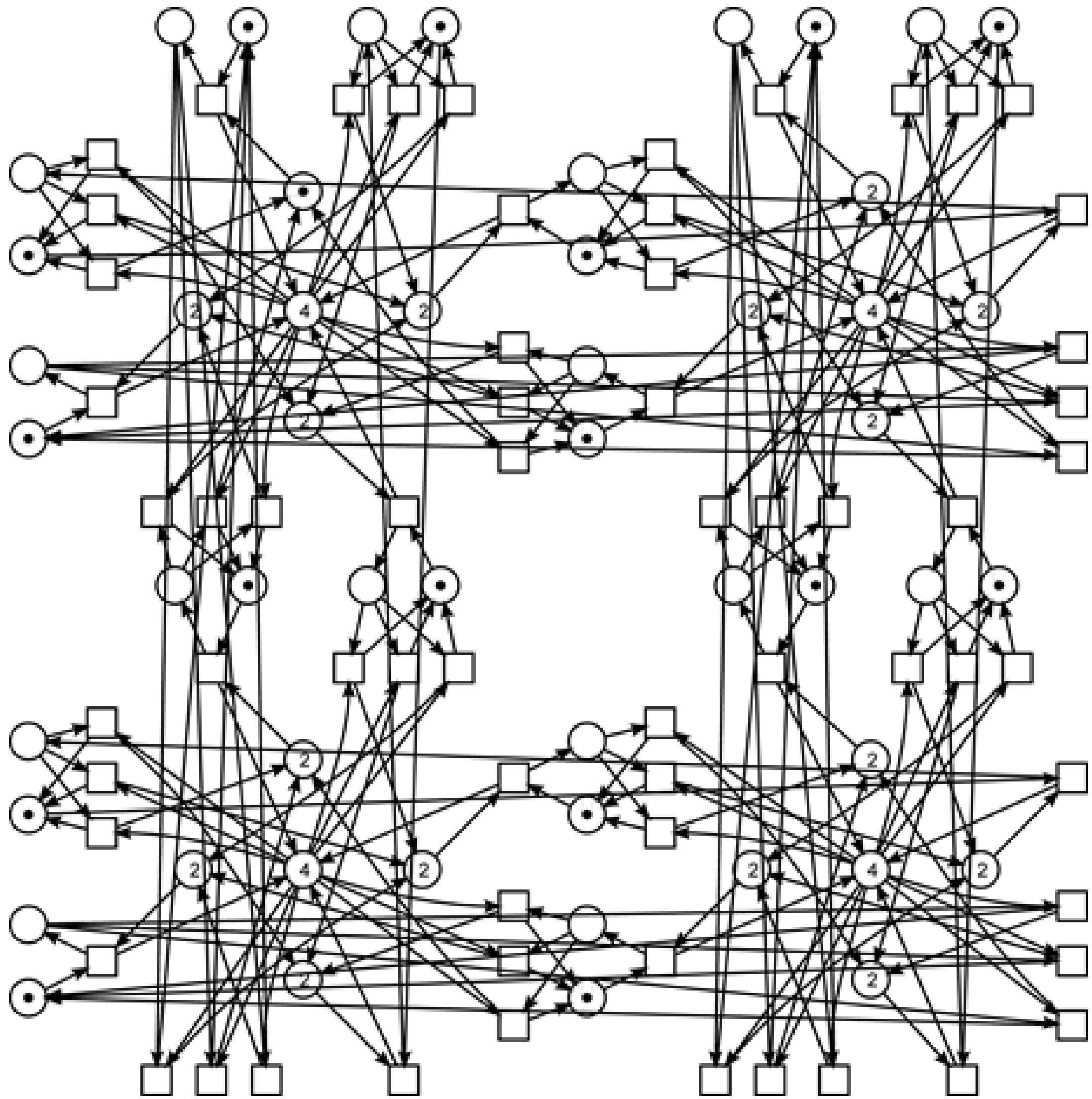
Load balancing

- Dynamic on demand – appoint a clan to a free node (version 1.1)
- Static – aggregate minimal clans into big clans according to the number of available computing nodes
- Hybrid – pre-aggregation to equal size and then dynamic scheduling clans to nodes (version 1.2)

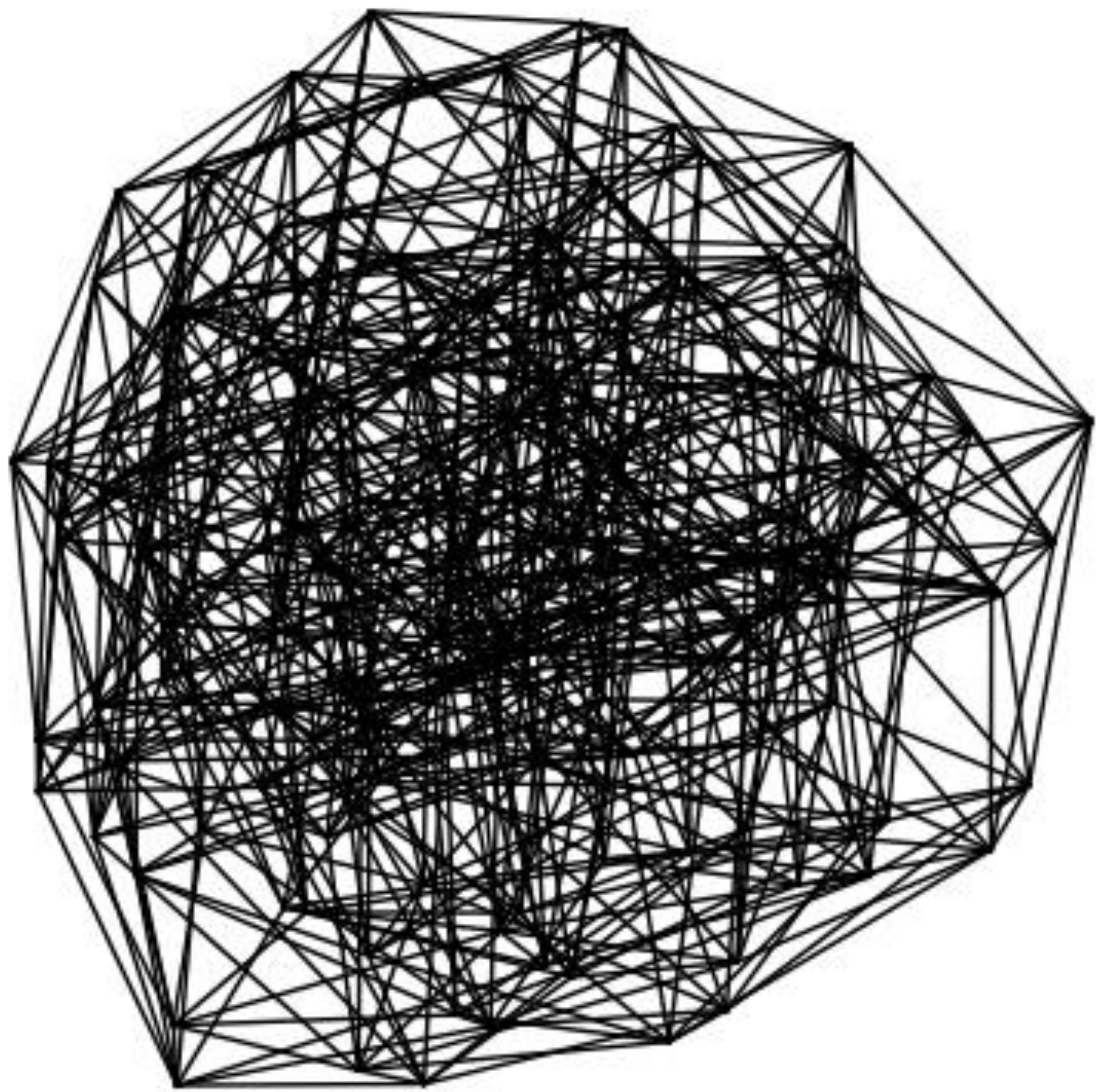
Aggregation idea



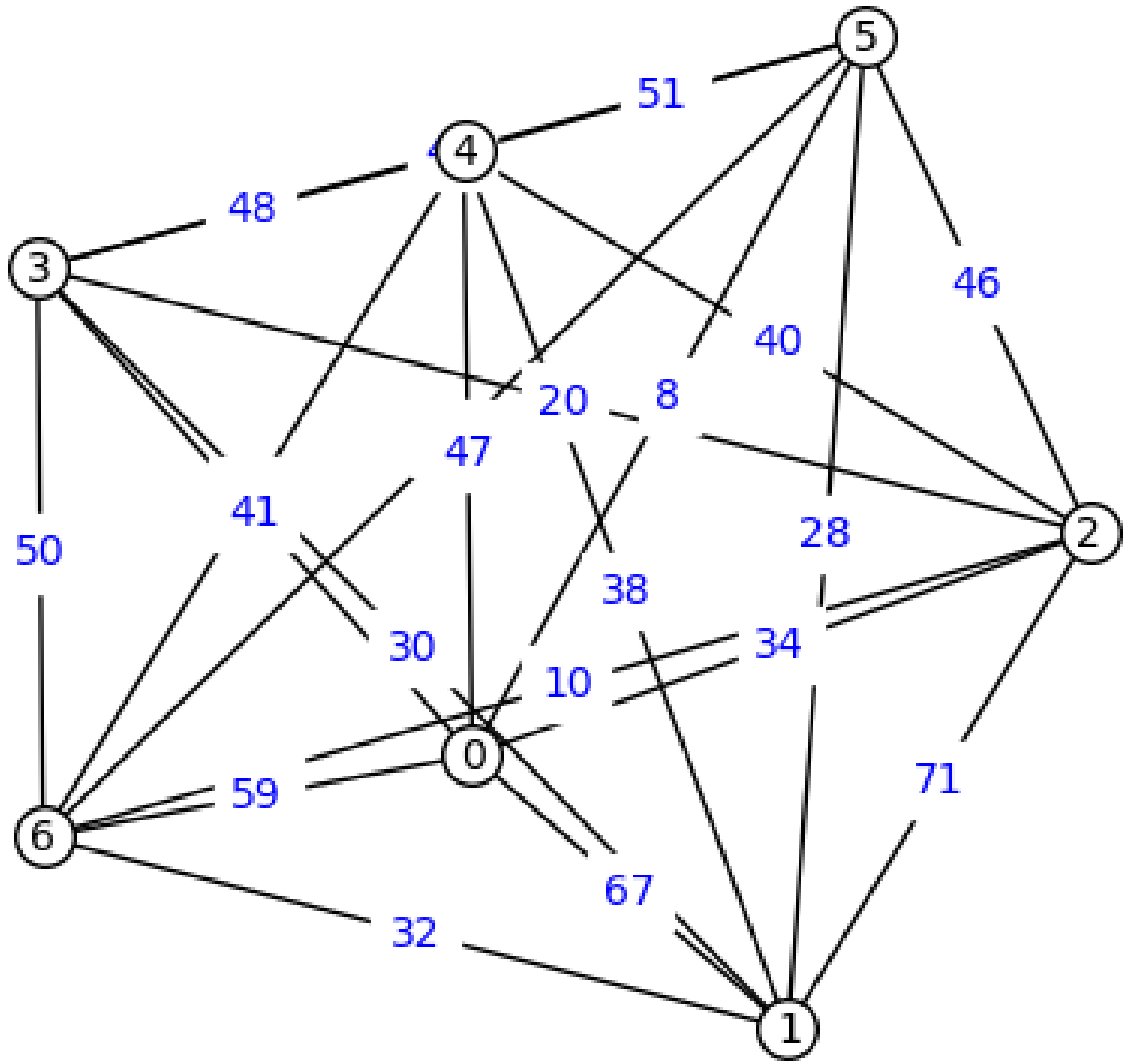
Hypertorus switching grid, 2D, 2x2 model



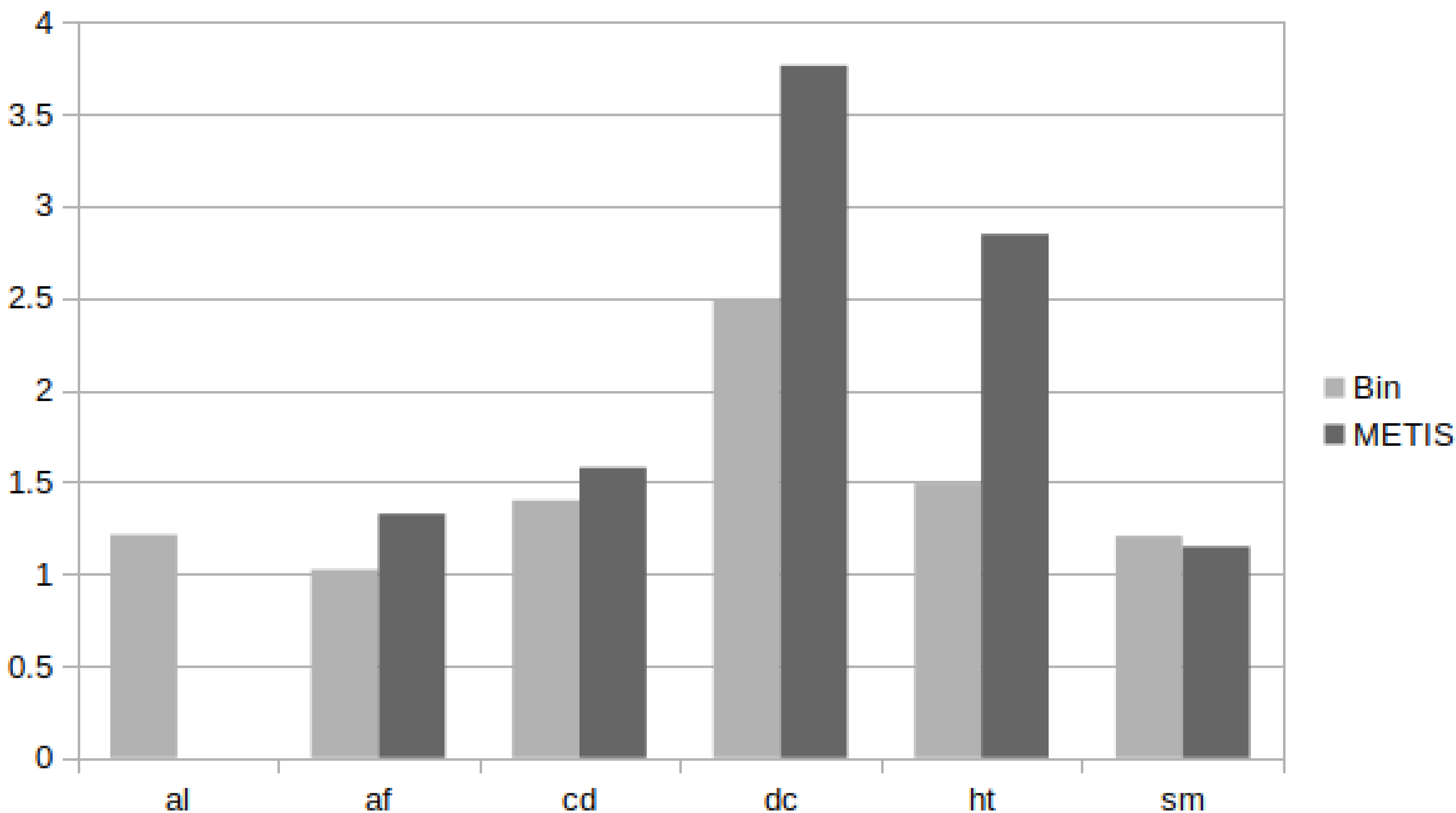
Decomposition of hypertorus 4D, 3x3



Aggregation by METIS into 7 clans



Extra speed-up because of aggregation



Conclusions

- Additional modules for ParAd:

a) clans aggregation with METIS;

b) clans aggregation with bin pack

- Extra speed-up because of aggregation up to 4 times
- Intractable tasks become tractable with clans composition on parallel architectures

http://daze.ho.ua

Recent references

- ParAd, https://github.com/dazeorgacm/ParAd
- Dmitry Zaitsev, Stanimire Tomov, Jack Dongarra. Solving Linear Diophantine Systems on Parallel Architectures, IEEE Transactions on Parallel and Distributed Systems, 30(5), 2019.
- Zaitsev D.A. Sequential composition of linear systems' clans, Information Sciences, Vol. 363, 2016, 292–307.

http://daze.ho.ua